*International Journal of Theoretical Physics, Vol. 29, No. 6, 1990*

# Painlevé Test and Energy Level Motion

**W.-H. Steeb, S. J. M. Brits,[1] and N. Euler[1]**

*Received October 11, 1988; revised November 22, 1988*

From the eigenvalue $H_\lambda|\psi_n(\lambda)\rangle = E_n(\lambda)|\psi_n(\lambda)\rangle$, where $H_\lambda \equiv H_0 + \lambda V$, one can derive an autonomous system of first-order differential equations for the eigenvalues $E_n(\lambda)$ and the matrix elements $V_{mn}(\lambda)$, where $\lambda$ is the independent variable. We perform a Painlevé test for this system and discuss the connection with integrability. It turns out that the equations of motion do not pass the Painlevé test, but a weaker form. The first integrals are polynomials and can be related to the Kowalewski exponents.

Various authors (Pechukas, 1983; Yukawa, 1985, 1986; Steeb, 1988; Steeb and Louw, 1986, 1987; Steeb and van Tonder, 1987*a*,*b*; Steeb *et al.*, 1988; Nakamura and Lakshmanan, 1986; Aizu, 1963) discuss the "motion" of energy levels $E_n(\lambda)$, where $\lambda$ plays the role of the time. Let us assume that the eigenfunctions are real orthogonal. Using the orthogonality relation

$$\langle \psi_m(\lambda)|\psi_n(\lambda)\rangle = \delta_{mn} \tag{1}$$

the completeness relation

$$1 = \sum_{n \in I} |\psi_n(\lambda)\rangle\langle\psi_n(\lambda)| \tag{2}$$

$$\left\langle \psi_n(\lambda)\middle|\frac{d\psi_n(\lambda)}{d\lambda}\right\rangle = 0 \tag{3}$$

and the assumptions that the eigenvalues are nondegenerate for $\lambda \geq 0$, these authors derive the following autonomous system of first-order ordinary

[1]Department of Applied Mathematics and Nonlinear Studies, Rand Afrikaans University, Johannesburg 2000, South Africa.



0020-7748/90/0600-0637$06.00/0 

differential equations:

$$\frac{dE_n}{d\lambda} = p_n$$

$$\frac{dp_n}{d\lambda} = 2 \sum_{m(\neq n)} \frac{V_{mn} V_{nm}}{E_n - E_m} \tag{4}$$

$$\frac{dV_{mn}}{d\lambda} = \sum_{k(\neq m,n)} \left[ V_{mk} V_{kn} \left( \frac{1}{E_m - E_k} + \frac{1}{E_n - E_k} \right) \right] - \frac{V_{mn}(p_m - p_n)}{E_m - E_n}$$

where $p_n(\lambda) = \langle \psi_n(\lambda) | V | \psi_n(\lambda) \rangle$ and $V_{mn}(\lambda) := \langle \psi_m(\lambda) | V | \psi_n(\lambda) \rangle (m \neq n)$. Notice that equation (3) is no longer true if $|\psi_m(\lambda)\rangle$ is complex orthogonal. If we have a finite-dimensional system with $N$ energy levels, then the number of differential equations $n$ is given by $n = N + N + N(N-1)/2 \equiv N(3/2 + N/2)$.

Pechukas (1983) and Yukawa (1985, 1986) discussed the dynamical system (4) in connection with quantum chaos [compare Steeb and Louw (1986) and references therein]. Moreover, Yukawa (1986) showed that the system (4) admits a Lax representation and is completely integrable. Consequently, no chaotic behavior can be expected for system (4). Nakamura and Lakshmanan (1984) gave the equations of motion for the eigenfunctions, namely

$$\frac{d|\psi_n\rangle}{d\lambda} = \sum_{m(\neq n)} \frac{V_{mn}}{E_n - E_m} |\psi_m\rangle \tag{5}$$

Steeb and Van Tonder (1987*a*,*b*) described the connection with the perturbation theory and considered the extended case $H_\lambda = H_0 + \lambda_1 V_1 + \lambda_2 V_2$. Steeb and Louw (1987) discussed energy-dependent constants of motion for system (4). The dependence of the survival probability as well as some thermodynamic quantities (free energy, entropy, specific heat) on $\lambda$ has been discussed by Steeb (1988). Let us mention that Aizu (1963) described in detail the parameter differentiation of quantum mechanical linear operators more than 25 years ago. The results given above can be considered as a straightforward application of his results. A survey on energy level motion is given by Steeb *et al.* (1988). Furthermore, we mention that the system given above is related to the generalized Calogero Moser system (Nakamura and Lakshmanan, 1986; Gibbons and Hermsen, 1984).

The question of whether or not energy levels can cross was first discussed by Hund (1927). He studied examples only and conjectured that, in general, no crossing of energy levels can occur. Von Neumann and Wigner (1929) investigated this question more rigorously and found the following theorem: Real symmetric matrices (respectively the Hermitian matrices)

with a multiple eigenvalue form a real algebraic variety of codimension 2 (respectively 3) in the space of all real symmetric matrices (respectively all Hermitian matrices). This implies the famous "no-crossing rule," which asserts that a "generic" one-parameter family of real symmetric matrices (or two-parameter family of Hermitian matrices) contains no matrix with a multiple eigenvalue. We emphasize that this no-crossing rule has only been proved for finite-dimensional matrices. "Generic" means that if the Hamiltonian admits symmetries, the underlying Hilbert space has to be decomposed into the invariant subspaces.

We perform a Painlevé test for system (4) and discuss the connection with integrability [compare Steeb and Louw (1986) and references therein]. System (4) is now considered in the complex domain.

First we consider the case with two energy levels $E_0(\lambda)$ and $E_1(\lambda)$. Then system (4) simplifies to

$$\begin{aligned} \frac{dE_j}{d\lambda} &= p_j \\ \frac{dp_0}{d\lambda} &= 2\frac{V_{01}^2}{E_0 - E_1} \\ \frac{dp_1}{d\lambda} &= 2\frac{V_{01}^2}{E_1 - E_0} \\ \frac{dV_{01}}{d\lambda} &= -\frac{V_{01}(p_o - p_1)}{E_0 - E_1} \end{aligned} \tag{6}$$

where $j = 0, 1$ and $V_{01} = V_{10}$. The form of system (6) motivates the introduction of the following quantities: $E := E_1 - E_0$, $p := p_1 - p_0$, and $V := V_{01}$. Then system (6) takes the form

$$\begin{aligned} \frac{dE}{d\lambda} &= p \\ \frac{dp}{d\lambda} &= 4\frac{V^2}{E} \\ \frac{dV}{d\lambda} &= -\frac{V_p}{E} \end{aligned} \tag{7}$$

Let us now perform the Painlevé test for system (7). The system is considered in the complex domain. Inserting the ansatz

$$\begin{aligned} E(\lambda) &\propto E^{(0)}(\lambda - \lambda_0)^m \\ p(\lambda) &\propto p^{(0)}(\lambda - \lambda_0)^n \\ V(\lambda) &\propto V^{(0)}(\lambda - \lambda_0)^q \end{aligned} \tag{8}$$

and comparing the exponents yields $n = q = m - 1$, where $m$ is arbitrary at this stage. Obviously, all terms in system (7) are dominant. Therefore, system (7) is scale invariant under

$$\begin{aligned} \lambda &\to \varepsilon^{-1}\lambda \\ E &\to \varepsilon^{-m}E \\ p &\to \varepsilon^{-m+1}p \\ V &\to \varepsilon^{-m+1}V \end{aligned} \tag{9}$$

Next we determine the coefficients $E^{(0)}$, $p^{(0)}$, and $V^{(0)}$. Requiring that $E^{(0)}, p^{(0)}, V^{(0)} \neq 0$, we find $m = 1/2$, $n = q = -1/2$. Furthermore, $p^{(0)} = E^{(0)}/2$ and $V^{(0)} = \pm iE^{(0)}/4$ with $E^{(0)}$ arbitrary. Consequently,

$$\begin{aligned} E(\lambda) &= E^{(0)}(\lambda - \lambda_0)^{1/2} \\ p(\lambda) &= \frac{E^{(0)}}{2}(\lambda - \lambda_0)^{-1/2} \\ V(\lambda) &= \mp\frac{iE^{(0)}}{4}(\lambda - \lambda_0)^{-1/2} \end{aligned} \tag{10}$$

is a solution of system (7), where $E^{(0)}$ and $\lambda_0$ are arbitrary. However, it is not the general solution which requires three arbitrary constants. When we determine the resonances [see Ablowitz *et al.* (1980) for the definition], using solution (10), we obtain $-1, 0, 1$. When we determine the Kowalewski exponents [see Yoshida (1983*a*,*b*) for the definition], using solution (10), we also find $-1, 0, 1$. The Kowalewski exponents 0, 1 can be associated with the (polynomial) first integrals of system (7). On inspection, we find that

$$\begin{aligned} I_1(E, p, V) &= \frac{p^2}{4} + V^2 \\ I_2(E, p, V) &= EV \end{aligned} \tag{11}$$

are first integrals. Then we obtain the scaling behavior

$$\begin{aligned} I_1(\varepsilon^{-1/2}E, \varepsilon^{1/2}p, \varepsilon^{1/2}V) &= \varepsilon^1 I_1(E, p, V) \\ I_2(\varepsilon^{-1/2}E, \varepsilon^{1/2}p, \varepsilon^{1/2}V) &= \varepsilon^0 I_2(E, p, V) \end{aligned} \tag{12}$$

where the exponents of $\varepsilon$ give the Kowalewski exponents, namely 0 and 1. Thus, the system (7) is algebraic completely integrable. For a detailed discussion of the definition "algebraic completely integrable" we refer to

van Moerbeke (1988). Inserting the expansion

$$E(\lambda)=(\lambda-\lambda_0)^{1/2}\sum_{j=0}^{\infty}E^{(j)}(\lambda-\lambda_0)^j$$

$$p(\lambda)=(\lambda-\lambda_0)^{-1/2}\sum_{j=0}^{\infty}p^{(j)}(\lambda-\lambda_0)^j \tag{13}$$

$$V(\lambda)=(\lambda-\lambda_0)^{-1/2}\sum_{j=0}^{\infty}V^{(j)}(\lambda-\lambda_0)^j$$

into system (7), we find one more arbitrary constant (besides $E^{(0)}$ and $\lambda_0$), so that expansion (13) represents the general solution. System (7) does not pass the Painlevé test, due to the dominant behavior. However, it passes the so-called quasi-Painlevé test, i.e., it admits an expansion of the form (13) with three arbitrary constants [see Steeb and Euler (1988) for more details].

Obviously, we can also find the general real solution to equation (7), namely

$$E^2(\lambda)=\frac{I_2^2}{I_1}+4I_1(\lambda-\lambda_0)^2$$

$$p^2(\lambda)=\frac{16I_1^3(\lambda-\lambda_0)^2}{I_2^2+4I_1^2(\lambda-\lambda_0)^2} \tag{14}$$

$$V^2(\lambda)=\frac{I_1I_2^2}{I_2^2+4I_1^2(\lambda-\lambda_0)^2}$$

The general solution contains three free parameters $I_1$, $I_2$, and $\lambda_0$ which are determined from the initial values $E(\lambda=0)$, $p(\lambda=0)$, and $V(\lambda=0)$.

For a system with three energy levels, i.e., $N=3$, we obtain an autonomous system with nine autonomous first-order differential equations. Introducing $E_{mn}:=E_m-E_n$ and $p_{mn}:=p_m-p_n$, we can reduce the number of equations to seven. Again we find that the system does not pass the Painlevé test, but the quasi-Painlevé test. In this case, too, we find that the system is algebraic completely integrable.

The general case can be discussed as follows: Let us assume that the Hamiltonian $H_\lambda=H_0+\lambda V$ acts in a finite-dimensional Hilbert space $\mathcal{H}$. The eigenvalues of $H_\lambda$ satisfy the characteristic equation

$$\det(H_\lambda-E)=0 \tag{15}$$

This is an algebraic equation

$$P(\lambda, E)\equiv A_0(\lambda)E^N+A_1(\lambda)E^{N-1}+\cdots+A_N(\lambda)=0 \tag{16}$$

of $E$ of degree $N = \dim \mathcal{H}$, with coefficients $A_j(\lambda)$ which are polynomials in $\lambda$. The coupling strength $\lambda$ is now considered in the complex domain. The equation $P(\lambda, E) = 0$ defines a curve in the two-dimensional complex projective space with inhomogeneous coordinates $\lambda$, $E$. Consequently, the functions $A_j(\lambda)$ are holomorphic in $\lambda$. It follows from function theory (Baumgärtel, 1964) that the roots of equation (15) are branches of analytic functions in $\lambda$ with only algebraic singularities. Consequently, system (4) passes the quasi-Painlevé test. The number of eigenvalues of $H_\lambda$ is a constant $k$ independent of $\lambda$, with the exception of some special values of $\lambda$. There are only a finite number of such exceptional points. Each level repulsion is associated with an exceptional point. At the exceptional points the two levels taking part in the repulsion coalesce. The closer the two levels approach each other for real values of the coupling strength, the nearer are the exceptional points to the real axis.

## REFERENCES


Ablowitz, M. J., Ramani, A., and Segur, H. (1980). *Journal of Mathematical Physics*, **21**, 715.

Aizu, K. (1963). *Journal of Mathematical Physics*, **4**, 762.

Baumgärtel, H. (1964). *Mathematische Nachrichten*, **26**, 361.

Gibbons, J., and Hermsen, T. (1984). *Physica*, **11D**, 337.

Hund, F. (1927). *Zeitschrift für Physik*, **40**, 742.

Nakamura, K., and Lakshmanan, M. (1986). *Physical Review Letters*, **57**, 1661.

Pechukas, P. (1983). *Physical Review Letters*, **51**, 943.

Steeb, W.-H. (1988). In *Finite Dimensional Integrable Nonlinear Dynamical Systems*, P. Leach and W.-H. Steeb, eds., World Scientific, Singapore.

Steeb, W.-H., and Euler, N. (1988). *Nonlinear Evolution Equations and Painlevé Test*, World Scientific, Singapore.

Steeb, W.-H., and Louw, J. A. (1986). *Chaos and Quantum Chaos*, World Scientific, Singapore.

Steeb, W.-H., and Louw, J. A. (1987). *Journal of the Physical Society of Japan*, **56**, 3082.

Steeb, W.-H., and van Tonder, A. J. (1987*a*). *Zeitschrift für Naturforschung*, **42a**, 819.

Steeb, W.-H., and van Tonder, A. J. (1987*b*). *South African Journal of Science*, **83**, 477.

Steeb, W.-H., van Tonder, A. J., Villet, C. M., and Brits, S. J. M. (1988). *Foundations of Physics Letters*, **1**, 147.

Van Moerbeke, P. (1988). In *Finite Dimensional Integrable Nonlinear Dynamical Systems*, P. Leach and W.-H. Steeb, eds., World Scientific, Singapore.

Von Neumann, J., and Wigner, E. (1929). *Physikalische Zeitschrift*, **30**, 467.

Yoshida, H. (1983*a*). *Celestial Mechanics*, **31**, 363.

Yoshida, H. (1983*b*). *Celestial Mechanics*, **31**, 381.

Yukawa, T. (1985). *Physical Review Letters*, **54**, 1883.

Yukawa, T. (1986). *Physics Letters A*, **116**, 227.